2024 Senior Regionals LIG
Lightning Round Answers

- Runner should change every 20 minutes - exercise is good for all!
- Runner always sets off to the back of their column of teams FIRST.
- Correct attempt 1=3 points. Correct attempt 2=2 points. 1 point thereafter.
- Teams may pass after at least 3 attempts, that question cannot then be returned to.
- To determine rank when final points are tied, the number of passes is considered.

PRACTISE QUESTION MUST BE ANSWERED CORRECTLY: **60 minutes**

1	4
2	$\frac{1+\sqrt{5}}{2}$
3	60
4	$\frac{28}{3}$
5	10 and -6
6	3
7	$24\sqrt{2}$
8	2
9	512
10	$4\sqrt{13}$
11	$\frac{2\pi}{3}-\frac{\sqrt{3}}{2}.$
12	$2-\sqrt{2}$
13	$\frac{1}{2}$
14	282
15	2008
16	$2\sqrt{5}-2\sqrt{3}$
17	-50
18	25

19	$(1,2,3)$
20	$\frac{\sqrt[3]{3}}{2}$
21	12
22	$\sqrt[4]{2}$
23	$-\frac{1}{4}$
24	$\frac{1}{8}\sin(8)-1.$
25	3
26	38
27	39
28	$\frac{11}{16}$
29	$\sqrt{82}$
30	$\frac{2}{7}$
31	$2e^2+2$
32	$\frac{1}{55}$
33	$\frac{1}{12}$
34	3
35	252

2024 Senior Regionals LIG Lightning Round Solutions

#1. Solution:
$f'(x) = 2x + b.\ f'(4) = 8 + b.\ f(4) = 16 + 4b - 20 = -4 + 4b. 8 + b = -4 + 4b. 12 = 3b.\ b = 4.$

#2. Solution:
$x^4 - 3x^2 + 2 - x^3 + 2x = 0\ (x^2 - 1)(x^2 - 2) - x(x^2 - 2) = 0\ (x^2 - 2)(x^2 - x - 1) = 0$
$x = \pm\sqrt{2}$ or $x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$. The largest is $\frac{1+\sqrt{5}}{2}$.

#3. Solution:
Since 56 is not a multiple of 3, the median is not 56. Therefore, $x < 56.$ Mean = $\frac{24 + 27 + 56 + 63 + x}{5} = \frac{170 + x}{5}$. Therefore x must be a multiple of 5. If x is the median, then $27 \leq x < 56$. The only possibilties are $x = 30$ or $x = 45$ as x needs to be a multiple of 3 too. If $x = 30,$ then the mean is not prime. If $x = 45$, then the mean is 43 which is prime. If 27 is the median, then the possibilities for x are $5, 10, 15, 20,$ and $25.$ If $x = 15$, then the mean is 37 (prime). If $x = 25$, then the mean is 39 (not prime). The other 3 possibilities also do not give a prime mean. So the 2 possibilities for x are 45 and $15.$ $45 + 15 = 60.$

#4. Solution:
$f(2) + 2 \cdot f(4) = 4.\ f(4) + 2 \cdot f(2) = 16$ or $f(4) = 16 - 2 \cdot f(2).$
$f(2) + 2 \cdot f(4) = f(2) + 2[16 - 2 \cdot f(2)] = 4\cdot f(2) + 32 - 4 \cdot f(2) = 4\cdot -3 \cdot f(2) = -28\cdot f(2) = \frac{28}{3}.$

#5. Solution:
If P is the point of tangency, then $P = (a, a^2 + 2a + 6)$. The line lintesects the $y-$ axis at the point $(0, -10)$. $y' = 2x + 2$, so $2a + 2 = b$, if b is the slope of the curve at point P. Between the point P and the y intersction of the curve, the slope of the tangent line is $b = \frac{a^2 + 2a + 6 - (-10)}{a} = \frac{a^2 + 2a + 16}{a}.\ 2a + 2 = \frac{a^2 + 2a + 16}{a}.$
$2a^2 + 2a = a^2 + 2a + 16. a^2 = 16$ or $a = \pm 4. b = 2(4) + 2 = 10$ or $b = 2(-4) + 2 = -6.$

#6. Solution:
If $\sqrt{a} - b$ is a root, then so is $-\sqrt{a} - b.$ We know that the sum of the roots of $x^2 + ax - b = 0$ is $-\frac{a}{1} = -a$ and the product is -b. $(\sqrt{a} - b) + (-\sqrt{a} - b) = -2b$ and $(\sqrt{a} - b)(-\sqrt{a} - b) = -a + b^2.$ So $-2b = -a$ and $-a + b^2 = -b.\ -2b + b^2 = -b. b^2 = b. b = 1$ and $a = 2. a + b = 3.$

#7. Solution:
It is first assumed that the rectangle has sides x, y, where the diagonal is 12. Using pythagorus, $x^2 + y^2 = 144$. Also, $xy = 72$. Solving these two equations yields x = y = $6\sqrt{2}$, so the perimeter is $24\sqrt{2}$.

#8. Solution:
$$\sqrt{4x^2 + 8x} - 2x = \left(\sqrt{4x^2 + 8x} - 2x\right)\frac{\sqrt{4x^2 + 8x} + 2x}{\sqrt{4x^2 + 8x} + 2x} = \frac{4x^2 + 8x - 4x^2}{\sqrt{4x^2 + 8x} + 2x} = \frac{8x}{\sqrt{4x^2 + 8x} + 2x}$$
$$= \frac{8}{\sqrt{4 + \frac{8}{x}} + 2}\ \lim_{x \to \infty} = \frac{8}{2 + 2} = 2\cdot$$

#9. Solution:
$v(t) = x'(t) = 20t^3 - 5t^4 = 5t^3(4-t)$ The particle is going to the right from $t = 0$ to $t = 4$. It is going to the left from $t = 4$ to $t = 5$. $x(0) = 0, x(4) = 256,$ and $x(5) = 0.$ Therefore$256 + 256 = 512$.

#10. Solution:
Let $x = \angle(ABD)$ and $y = \angle(DBC)$. Since triangles ABD and BDC are isosceles, $\angle(C) = y$ and $\angle(A) = x$. $x + x + y + y = 180$ or $x + y = 90$. We use the Pythagorean Theorem to find $AB = 4\sqrt{13}$

#11. Solution:
Recognize that the integral represents a circle drawn with centre $(0, 0)$, and radius of 2, between $x = 1$ and $x = 2$. When $x = 1, y = \sqrt{3}$, a right angle triangle can be drawn with height $\sqrt{3}$ and base of 1, and an angle of 60 degrees = one sixth of a circle. The region of the integral is one-sixth of a circle with radius 2 minus the area of a triangle with legs 1 and $\sqrt{3}$. $A = \frac{1}{6}(4\pi) - \frac{1}{2} \cdot 1 \cdot \sqrt{3} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$.

#12. Solution:
$a, b,$ and c are the roots of $x^3 - 3x^2 - 2x + 2 = 0$. -1 is a root, so $(x+1)$ is a factor. The other factor is $x^2 - 4x + 2$. Using the Quadratic Formula, we get $x = 2 \pm \sqrt{2}$. The 3 roots are $-1,$ and $2 - \sqrt{2},$ and $2 + \sqrt{2}$. $b = 2 - \sqrt{2}$.

#13. Solution:
Through trial and error on known angles between $0 < x < 1$. $5^{1-4\cdot 0.5^2} = 5^0 = \sin\left(\frac{\pi}{2}\right)$.

#14. Solution:
One side of the pentagon is the hypotenuse of a right triangle. The only possibility for this hypotenuse is 15, because 9, 12, 15 is a Pythagorean Triple. The lengths of the sides if the rectangle are $16(9+7), 21(12+9), 16,$ and 21. Area = $16 \cdot 21 - 0.5 \cdot 12 \cdot 9 = 336 - 54 = 282$.

#15. Solution:
Let $y = x + 5$. Now the function is
$z = (y-3)((y-1)(y+1)(y+3) + 2024 = (y^2-9)(y^2-1) + 2024 = y^4 - 10y^2 + 2033.$
$z' = 4y^3 - 20y$ To find the minimum, $z' = 0 = (4y)(y^2 - 5)$ Critical points are $y = 0$ or $y = \pm\sqrt{5}$. Evaluating z for the values of y, we get 2033 and 2008. The minimum value is 2008.

#16. Solution:
The path for the flying bug is the hypotenuse of a right triangle. Distance is $2^2 + (2\sqrt{2})^2 = x^2$ or $x = 2\sqrt{3}$. For walking bug, open up the net for the cube and see that $4^2 + 2^2 = y^2$ or $y = 2\sqrt{5}$. We need the difference, so the answer is $2\sqrt{5} - 2\sqrt{3}$.

#17. Solution:
The common difference is $2a - a = a$. Therefore $b - 2a = a$ and $b = 3a$. The 4th term is $a - 6 - b = a - 6 - 3a = 4a$ or $-2a - 6 = 4a$ or $a = -1$. The sequence is: $-1, -2, -3, -4, \ldots$ The 50th term is -50.

#18. Solution:
$-\frac{1}{4}x^2 + 5 = x^2 - k$. $5 + k = \frac{5}{4}x^2$. So $k + 5 \geq 0$ or $k \geq -5$. $5 + k = \frac{5}{4}x^2$ or $\frac{4}{5}(5+k) = x^2$.
$y = x^2 - k = \frac{4}{5}(5+k) - k = 4 + \frac{4}{5}k - k = 4 - \frac{1}{5}k$. Since $y > 0$, we have $4 - \frac{1}{5}k > 0$ or $k < 20$.
Combining these results, we have $-5 \leq k < 20$. There are 25 integers in this interval.

#19. Solution:
Let $x = a, y = b - 1, \text{ and } z = c - 2$. (1): $x + \log(x) = y$ (2) $y + \log(y) = z$ (3) $z + \log(z) = x$ Obviously $(x, y, z) = (1, 1, 1)$ is a solution. Case1: $x > 1$ From (1): $x < y$ From (2): $y < z$ From (3): $z < x$ This is impossible. Case 2: $0 < x < 1$ From (1), $x > y$ From (2): $y > z$ From (3),$z > x$. This is impossible, so the only answer is $(x, y, z) = (1, 1, 1) \text{ or } (a, b, c) = (1, 2, 3)$

#20. Solution:
$y' = 3x^2 + 3'$, so the slope of the tangent line is $3a^2 + 3$. The slope is also $\frac{a^3 + 3a + 3}{a}$, so
$3a^2 + 3 = \frac{a^3 + 3a + 3}{a}$. $3a^3 + 3a = a^3 + 3a + 3$. $2a^3 = 3$. $a = \frac{\sqrt[3]{3}}{2}$.

#21. Solution:
$\triangle ABC$ similar to $\triangle DBA'$ so $\frac{4}{5} = \frac{AD}{3}$ or $AD = \frac{12}{5}$. Also $\triangle DAE$ similar to $\triangle CBA$ so $\frac{DE}{AD} = \frac{AC}{BC}$ or $\frac{DE}{AD} = \frac{4}{5}$. So each altitude is $\frac{4}{5}$ of the previous altitude. The sum becomes an infinite geometric series with $a = \frac{12}{5}$ and $r = \frac{4}{5}$. $S = \frac{a}{1 - r} = \frac{\frac{12}{5}}{1 - \frac{4}{5}} = 12$.

#22. Solution:
$x^2 - 2x - 8 = (x - 4)(x + 2)$ So $f = 0$ means $x = 4 \text{ or } x = -2$ However $-2^{\frac{1}{8}}$ is not a real root $f\left(4^{\frac{1}{8}}\right) = f(-2) = 0\,4^{\frac{1}{8}} = 2^{\frac{1}{4}} = \sqrt[4]{2}$. So the only answer is $\sqrt[4]{2}$.

#23. Solution:
$y = r\sin(\theta) = (\sin(\theta) + 1)(\sin(\theta)$. $= \sin^2(\theta) + \sin(\theta) = \left(\sin^2(\theta) + \sin(\theta) + \frac{1}{4}\right) - \frac{1}{4}$ Using complete the square where the format is $a(x + p)^2 + q$, and the minimum is defined by $-p, q$. $= \left(\sin(\theta) + \frac{1}{2}\right)^2 - \frac{1}{4}$. The minimum occurs when $\sin(\theta) = -\frac{1}{2}$. Substituting back into $r = \sin(\theta) + 1$, $r = \frac{1}{2}$, therefore the least value of y is $-\frac{1}{4}$.

#24. Solution:
$v(t) = \int(a(t))dt = \int\cos^2(2t)\,dt = \int\frac{\cos(4t) + 1}{2}dt = \frac{\sin(4t)}{8} + \frac{t}{2} + C$. Substitute $t = 0$ to get $C = -2$.
Velocity function is $v(t) = \frac{\sin(4t)}{8} + \frac{t}{2} - 2$. $v(2) = \frac{\sin(8)}{8} - 1 = \frac{1}{8}\sin(8) - 1$.

#25. Solution:
If $x^2 - 4 = 1$, Then $x = \pm\sqrt{5}$. If $x^2 - 2x = 0$, then $x = 0 \text{ or } 2$. However, $x = 2$ gives 0^0 which is undefined. So, there are 3 solutions.

#26. Solution:
The interior angles are all equal to $108°$, so angle $A = 108°$. Triangle ABE is isosceles, so angle $AEB =$ angle $ABE = 36°$. Using the Law of Sines: $\frac{\sin(36)}{AB} = \frac{\sin(108)}{BE}$, or $\frac{\sin(36)}{AB} = \frac{\sin(72)}{BE}$.
$\frac{BE}{AB} = \frac{\sin(72)}{\sin(36)} = \frac{\sin(2 \cdot 36)}{\sin(36)} = \frac{2 \cdot \sin(36) \cdot \cos(36)}{\sin(36)} = 2 \cdot \cos(36)$. $2 + 36 = 38$.

#27. Solution:
$\triangle DME$ is similar to $\triangle DEC$ so $\frac{EM}{DM} = \frac{EC}{DE}$. $DE^2 = 2^2 + 3^2$, so $DE = \sqrt{13}$. $\frac{3}{2} = \frac{EC}{\sqrt{13}}$ or $EC = \frac{3\sqrt{13}}{2}$. Area of the rhombus $= \frac{1}{2} \cdot (2 \cdot \sqrt{13}) \cdot \left(2 \cdot \frac{3}{2} \cdot \sqrt{13}\right) = 39$.

#28. Solution:
Since Roger won the first set, there are 6 ways that he can win the match.
$P(WW) = \frac{1}{4}, P(LWW \text{ or } WLW) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}, P(WLLW \text{ or } LWLW \text{ or } LLWW) = 3 \cdot \frac{1}{16} = \frac{3}{16}$.
$\frac{1}{4} + \frac{1}{4} + \frac{3}{16} = \frac{11}{16}$.

#29. Solution:
Let $ABCD$ be the parallelogram with shorter diagonal $\overline{BD}$, and d is the longer diagonal. In Triangle ABD we have $8^2 = 3^2 + 8^2 - 2 \cdot 3 \cdot 8 \cdot \cos(A) \text{ or } \cos(A) = \frac{3}{16}$. In triangle ACD we have $d^2 = 3^2 + 8^2 - 2 \cdot 3 \cdot 8 \cdot \cos(D)$
The sum of any two adjacent angles in a parallelogram = 180 degrees, and $\cos(x) = -\cos(180 - x)$. Therefore $\cos(D) = -\cos(A)$. $d^2 = 3^2 + 8^2 - 2 \cdot 3 \cdot 8 \cdot \cos(D) = 9 + 64 + 48\cos(A) = 73 + 48 \cdot \frac{3}{16} = 82.$
$d = \sqrt{82}$

#30. Solution:
Consider a piece of wire of length 7. To solve, make a number line where point A is at 0, point B is at 7 and point C is at 6. Point P can fall anywhere between B and C. $\frac{BC}{AC} = \frac{1}{7}$. We must double the answer, since P can also fall between $0 \text{ and } 1$. $\frac{1}{7} + \frac{1}{7} = \frac{2}{7}$.

#31. Solution:
Let $y = \sqrt{x}$. So $y^2 = x \text{ and } 2ydy = dx$. Our integral becomes $2\int_0^2 (ye^y dy)$ Use integration by parts where $u = y \text{ and } dv = e^y dy$. $du = dy \text{ and } v = e^y$. $uv - \int vdu = ye^y - \int e^y dy = ye^y - e^y = e^y(y-1)$. So
$2\int_0^2 (ye^y dy) = 2[e^2 + e^0] = 2e^2 + 2$

#32. Solution:
The expression factors into $\left(1 + \frac{1}{\sqrt{n}}\right)\left(1 - \frac{1}{\sqrt{1+n}}\right)$. Expanding the product:
$$\left[\left(1+\frac{1}{1}\right) \cdot \left(1 - \frac{1}{\sqrt{2}}\right)\right]\left[\left(1+\frac{1}{\sqrt{2}}\right)\left(1 - \frac{1}{\sqrt{3}}\right)\right]\left[\left(1+\frac{1}{\sqrt{3}}\right)\left(1 - \frac{1}{\sqrt{4}}\right)\right]\left[\left(1+\frac{1}{\sqrt{4}}\right)\left(1 - \frac{1}{\sqrt{5}}\right)\right]..$$
$$= (1+1)\left[\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right)...\left(1-\frac{1}{99}\right)\right]\frac{9}{10} = 2\left[\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot ... \cdot \frac{98}{99}\right]\frac{9}{10} = 2 \cdot$$

#33. Solution:
For this solution, use $|ABC|$ to represent the area of triangle ABC. Triangle MAB is similar to triangle DHB, with ratio 2 to 1. $HB = \frac{1}{3}AH$, so $|HBD| = \frac{1}{3}|HAD| = \frac{1}{3}\left(\frac{1}{2}|DBC|\right) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$. Similarly, $|DET| = \frac{1}{12}$. $|BCED| = \frac{1}{4} - \frac{1}{12} - \frac{1}{12} = \frac{1}{12}$. Alternatively: The kite $BCED$ has height $\frac{1}{2}$, width x, and therefore the area $BCED = \frac{x}{4}$. (1) Let the triangles of MHB and AET have a height of $\frac{1-x}{2}$, therefore MHB and AET have areas $\frac{1-x}{4}$ each = $\frac{2-2x}{4}$. (2) Area MAD is $\frac{1}{2}$. (3) Triangles MBD and DET combined would have area of $\frac{1}{4}$ minus the area of the diamond = $\frac{1-x}{4}$. All the above areas should add up to 1, so we solve for x: $x = \frac{1}{3}$, therefore area $BCED$ is $\frac{1}{12}$.

#34. Solution:
Since $\sin(x) \leq 1 \text{ and } \log_{10} x < 1$, when $x < 10$, we need the number of times the 2 graphs cross between $0 \text{ and } 3\pi$. By plotting a log graph which intersects (1,0) and (10,1), and plotting the sine graph above it, we can see the graphs cross once between $0 \text{ and } \pi$ and twice between $2\pi \text{ and } 3\pi$.

#35. Solution:
Let m be the nuber of minutes after which Beyonce catches up to Kelly. Then $\frac{1}{12}m + 1 = \frac{1}{9}m \text{ or } m = 36$. Let n be the number of minutes after which Michelle catches up to Kelly. The $\frac{1}{14}n + 1 = \frac{1}{12}n \text{ or } n = 84$. The least common multiple of $36 \text{ and } 84$ is 252. Alternatively: Imagine that the track turns to the opposite direction that they are walking, and turning at a speed of 14 minutes per revolution, so Kelly does not move. Let the distance of the track = d, so that Michelle's effective speed of walking would be $\frac{d}{12} - \frac{d}{14} = \frac{d}{84}$. the time that Michelle takes to meet Kelly at the same point again (the original point as Kelly hasn't moved) = 84 minutes. Beyonce's effective speed = $\frac{d}{9} - \frac{d}{14} = 5\frac{d}{126}$. the time that Beyonce takes to meet Kelly at the same point again = $\frac{126}{5}$ minutes. The lowest common multiple of 84 and $\frac{126}{5}$ is 252 minutes for all three to be on the same point on the track again.